Qualifying Exam – Partial Differential Equations (II) – May 2019

INSTRUCTION: *Any resources or communications are NOT allowed during the exam. You may use all well-known results without proof unless you are asked to prove such a result, but you must indicate the result and include necessary steps. Work on familiar problems first.*

#1. Let $n \geq 2$ and $1 < p < n$ be given. Show that the inequality

$$\int_{\mathbf{R}^n} \frac{|u|^q}{|x|^r} dx \leq C \int_{\mathbf{R}^n} |Du|^p dx \quad \forall\, u \in W_0^{1,p}(\mathbf{R}^n),$$

where $C > 0$ is a constant, can only hold for certain q and r. Find such q and r, and **prove the inequality**.

#2. Let Ω be a bounded open set in $\mathbf{R}^n$. Show that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|u\|_{H^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)} \leq c_2 \|u\|_{H^2(\Omega)} \quad \forall\, u \in H_0^2(\Omega).$$

Furthermore, if Ω is a bounded domain with $\partial\Omega \in C^2$, show that the same result holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$. (*Henceforth, a domain is a connected open set.*)

#3. Let $p \geq 1$ and $\Omega \subset \mathbf{R}^n$ be an open set. Consider the following statements:
(a) If $u_k \to u$ in $W^{1,p}(\Omega)$, then $u_k^+ = \max\{u_k, 0\} \to u^+ = \max\{u, 0\}$ in $W^{1,p}(\Omega)$.
(b) If $u_k \to u$ and $v_k \to v$ in $W^{1,p}(\Omega)$, then $\max\{u_k, v_k\} \to \max\{u, v\}$ and $\min\{u_k, v_k\} \to \min\{u, v\}$ in $W^{1,p}(\Omega)$.
(c) If $g \in W^{1,p}(\Omega)$, $f \in W_0^{1,p}(\Omega)$ and $|g(x)| \leq f(x)$ a.e. in Ω, then $g \in W_0^{1,p}(\Omega)$.
Show that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c), and then **prove (a).**

#4. Let Ω be a bounded domain in $\mathbf{R}^n$ with $\partial\Omega \in C^1$ and let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Give a proper **definition of a weak solution** $u \in H^1(\Omega)$ to the Neumann's BVP

$$(*) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

so that, when f, g are smooth, all smooth classical solutions u on $\bar{\Omega}$ are a weak solution. Show that $(*)$ has a weak solution $u \in H^1(\Omega)$ if and only if $\int_\Omega f dx + \int_{\partial\Omega} g dS = 0$. Furthermore, if u_1, u_2 are two weak solutions to $(*)$, prove $u_1 = u_2 + C$ for a constant C.

#5. Let Ω be a bounded open set in $\mathbf{R}^n$, a_{ij}, b_i, $c \in L^\infty(\Omega)$ and assume

$$Lu = -\sum_{i,j=1}^n D_i(a_{ij}(x) D_j u) + \sum_{i=1}^n b_i(x) D_i u + c(x) u$$

is uniformly elliptic on Ω. Show that the BVP $\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$ has a unique weak solution $u \in H_0^1(\Omega)$ for each $f \in L^2(\Omega)$ if either (a) or (b) given below holds:

(a) $c(x) \geq 0$ a.e. in Ω. (b) $b_i \in W^{1,\infty}(\Omega)$, $c(x) \geq \frac{1}{2} \sum_{i=1}^n D_i b_i(x)$ a.e. in Ω.

#6. Let $\Omega = \{x \in \mathbf{R}^n : a < |x| < b\}$, where $b > a \geq 0$ and $n \geq 2$, and let $\sigma \colon \mathbf{R} \to \mathbf{R}$ be a given continuous function. Assume $U \in H_0^1(a, b)$ is a weak solution of

$$-(U'(r) r^{n-1})' = \sigma(U(r)) r^{n-1} \quad \text{in } (a, b).$$

Define $u(x) = U(|x|)$. Show that u belongs to $H_0^1(\Omega) \cap C^{0,\frac{1}{2}}(\bar{\Omega})$ and is a weak solution of

$$-\Delta u = \sigma(u) \quad \text{in } \Omega.$$